On Cubic Birational Space-Transformations.

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In 1871 Cremona* gave a method of determining all the birational transformations in which a given surface becomes a plane, when one representation of it on a plane is known; he added a great many examples of cubic transformations, but did not aim at completeness. For the non-singular cubic surface there exist seven transformations, which have been given by Loria† and Sturm.‡ This paper deals with singular cubic surfaces.

The method is briefly as follows. It is required to discover all the homaloidal families of cubic surface (ϕ_1) containing a given member ϕ, and having certain singularities in common with it. Suppose ϕ represented on a plane α by an auxiliary transformation T; all the cubic families here concerned have at least one common node, and T is taken to be projection from the first of these. Then to the family of curves of intersection of ϕ with (ϕ_1) there corresponds a homaloidal family of proper or degenerate curves in the plane α, satisfying certain conditions; and conversely, to every such plane family there corresponds a homaloidal family (ϕ_1), each member of which contains a certain fixed fundamental system H consisting of curves, simple points, points of contact of various orders and multiple points. The different plane homaloidal families of the earlier degrees are known,§ and by examining each in turn we are led back to all the existing families (ϕ_1) of the required kind.

The following investigation treats of cubic surfaces having 1, 2, 3, 4 nodes or a nodal line; there are further subdivisions according as the tangent cone at any node has a fixed part of order 0, 1 or 2, presenting 4, 7 or 9 conditions to the surface.

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* *Rend. R. Ist. Lomb.*, (2) IV, pp. 269, 315; cited as C. See also *Math. Ann.*, IV, p. 213.

† *Atti Acc. Sci. Torino*, XXVI, p. 197; cited as L.

‡ *Die Lehre v. d. geom. Verwandtschaften*, Leipzig, 1909, IV, p. 360; cited as S.

§ Cremona, *Mem. dell' Acc. di Bologna*, (2) V, p. 117. Cayley, *Proc. Lond. Math. Soc.*, III, p. 127 = *Coll. Math. Papers*, VII, p. 189.

I. *One Node p.*

(i) Let p be a cnicnode. Then the intersection of ϕ, ϕ_1 is a $C_9 p^4$.

[I follow Cremona in using a suffix to show the order of the curve C; the small letters which immediately follow to denote points on it, or numerals to denote the number of points; and an index to show the multiplicity of each point. Where it is necessary to distinguish between branches through a binode touching the fixed and variable planes respectively (see below), this is shown by a suffix to the small letter; *e. g.*, $C_5 p_f^2 p_v$ is a quintic curve with three branches through the binode p, two touching the fixed plane and one the variable plane.]

The projection of this $C_9 p^4$ is a plane $C_5\, abcdef$, where these points are the fundamental points of T, and are the traces on α of the six rays, or straight lines through p lying on ϕ; these six points lie on a conic, the trace on α of the tangent cone at p. This plane curve must be further restricted so as to become the generic curve of a homaloidal family of order 5 or less; the possible families are:

$$C_1, \quad C_2 3, \quad C_3 1^2 4, \quad C_4 3^2 3, \quad C_4 1^3 6, \quad C_5 6^2, \quad C_5 1^3 3^2 3, \quad C_5 1^4 8,$$

the figures showing the fixed base points of the families. When the family is of order less than 5, the C_5 degenerates into the variable member of the proper family and a fixed part passing through all of the six fundamental points of T which are not base points of the variable part.

There is a homaloidal family (ϕ_1) corresponding to each way of arranging this plane family and dividing the fundamental points of T between the fixed and variable parts; the other base points of the variable part give isolated points of the fundamental system H of (ϕ_1). If this variable part has a second branch through a fundamental point of T, then H contains the corresponding ray; the remaining fixed curves in H correspond to the fixed plane curves, and the former have as many branches through p as the latter have intersections (other than fundamental points) with the conic *abcdef*.

If one of the fixed plane curves is this conic, the surfaces (ϕ_1) contain all points adjacent to p on ϕ; *i. e.*, they have the same tangent cone; this possibility will be considered separately later.

If the fixed curves in H have three or more branches through p, then the existence of the node is a necessary consequence of the passage of the surface through these curves, and we have a particular case of a transformation of a non-singular surface, in which the fundamental curve has acquired a triple point. No further attention is here paid to such cases.

(ii) Let p be a binode. The conditions which a binode imposes on the coefficients of a general cubic surface are not linear unless one tangent plane at the node is fixed. Three rays lie in each plane; the C_9 of intersection of ϕ, ϕ_1 projects into a plane $C_4 def$, where d, e, f lie in the variable plane.

If the fixed curves in H have three or more branches touching the fixed plane at the node, then the passage of ϕ through H causes the degeneration of the tangent cone; if the fixed curves have two or more branches touching the other plane, this becomes fixed, giving a case treated below; the same happens if the straight line def in the projection is fixed.

The varieties of binode, called by Cayley B_4, B_5, B_6 (in which the edge, or intersection of tangent planes, is a ray), impose conditions which are not linear unless either the edge, or the other two rays in the fixed plane, be fixed. Either supposition leads merely to particular cases of other transformations.

(iii) Let p be a cnicnode with a fixed tangent cone. The projection of the intersection is any plane cubic. This includes the case of a binode with both tangent planes fixed, or any kind of unode.

II. *Two Nodes p, q.*

(i) Let p, q be cnicnodes. The intersection of ϕ, ϕ_1 consists of $pq + C_8 p^3 q^3$; the latter part projects into a plane $C_5 a^3 bcde$, for here two rays coincide in pq (the word "ray" meaning throughout a line through the first node p). Corresponding to each branch of a plane curve through a, there is a branch of a twisted curve through q. All cases in which the fixed curve has two branches through one node can be neglected as giving particular cases of previous transformations.

(ii) Let p be a binode. Since pq lies on the surface, it must lie on either the fixed or the variable tangent plane at p.

First let it lie on the variable plane. The projection is $C_4 a^3 b$, where a, a, b are the traces of the three rays in the variable plane, two of which coincide in pq. If now the curve in H has one branch through p touching the variable plane, this plane becomes fixed, and the whole tangent cone at p is fixed (see below). The line ab cannot be part of the plane curve, for the plane pab meets ϕ in pq, pq, pb only.

(iii) Next let p be a binode, and let q lie on the fixed tangent plane. Then this plane touches the surfaces all along pq, which counts twice in the intersection. The projection of the remaining C_7 is $C_4 a^2 def$. The tangent cone at q touches the fixed plane, and one other fixed curve through q would make

that node a necessity; therefore the fixed plane curve cannot pass through a. But one other fixed curve through p touching the fixed plane would cause the degeneration of the tangent cone at p. It follows that there is no fixed curve.

(iv) Let p be a cnicnode with a fixed tangent cone. Then the surfaces touch the cone all along pq, which counts twice in the intersection. The projection of the remaining C_7 is $C_3 a^2$. Again, the fixed part cannot pass through a.

(v) Let p, q both be binodes. One tangent plane at each node contains pq, and these must be the same plane, osculating the surfaces all along pq, counting three times in the intersection and causing the degeneration of each tangent cone.

(vi) Let p be a cnicnode with a fixed tangent cone, and let q be a binode. Then the fixed plane at q contains pq, which counts three times in the intersection. The surfaces touch but do not osculate the fixed plane all along pq. The projection of the remaining C_6 is $C_3 a^2$, with one fixed tangent at the node.

(vii) If both tangent cones are fixed, the projection is $C_3 a^3$; no such homaloidal family exists.

III. *Three Nodes p, q, r.*

(i) Let p, q, r be cnicnodes. The intersection consists of $pq + pr + qr + C_6 p^2 q^2 r^2$; the projection of the last part is $C_4 a^2 b^2 cd$, for here two pairs of rays coincide in pq, pr respectively. Since any fixed branch through any node makes that node a necessity, we need only consider the cases in which $a^2 b^2$ lie on the variable plane curve.

(ii) Let p be a binode. Then the fixed plane must contain one other node, say q, and the variable plane contains r. The intersection contains pr, qr and a line of contact pq, making the node q a necessity.

(iii) Let p be a cnicnode with a fixed tangent cone. Then the intersection contains qr and two lines of contact pq, pr, making the nodes q, r a necessity.

All other assignments of fixed planes or cones fail in the same way to give new transformations.

IV. *Four Nodes.*

The system is homaloidal without further condition, giving a particular case of transformation No. 2.

V. *A Nodal Line.*

A nodal straight line counts four times in the intersection. If the vertex of projection is any point p on the line, the remaining C_5 projects into $C_5 a^4 bc$, where b, c are the traces of the two generators through p. Or if T is Cremona's quadro-cubic transformation, the plane curve is $C_4 a^3$.

Various particular cases, but no new transformations, arise from considering the cases in which the simple directrix coincides with the nodal line, when there is contact of one or both of the sets of sheets with a fixed plane or quadric all along the line.

VI.

In the following list the transformations given by Cremona and Sturm are included for the sake of completeness. The transformations are arranged in the first place according to the order of the reverse transformation, secondly according to the number of nodes, and finally according to the degree of the fixed parts (if any) of the tangent cones.

No.	Order.	Isolated points in H.	Multiple points; curves in H; (authorities, see references at end of list).
1	3–2	none.	nodal line; 3 generators; (C., S., Cayley, Noether).
2	3–3	"	no node; C_6; (C., L., S., Cayley, Noether, De Paolis, Kantor, Berry, Beloch).
3	"	"	node; $C_6 p^2$; (C., S., Noether, De Paolis, Del Pezzo, Kantor).
4	"	"	node, fixed cone; $C_6 p^4$; (C.).
5	"	2 points.	nodal line; 2 generators; (C., S., De Paolis, Del Pezzo, Kantor, Berry).
6	3–4	1 point.	no node; C_5; (C., L., S.).
7	"	"	node; $C_5 p^2$; (Berry, Del Pezzo).
8	"	"	binode; $C_5 p_f^2 p_v$.
9	"	"	node, fixed cone; $C_3 p^2$ + 2 rays; (Berry).
10	"	"	node p, fixed cone, binode q; $C_2 pq_v$ + line of osculation pq, surfaces touch plane.
11	"	4 points.	nodal line; 1 generator; (C., S.).
12	3–5	2 points.	no node; $C_3 + C_1$; (C., L., S.).
13	"	"	node; $C_4 p$; (C.).
14	"	"	" $C_3 p$ + ray.
15	"	"	binode; $C_4 p_f^2$.
16	"	"	" $C_2 p_f$ + ray$_f$ + ray$_v$.
17	"	"	node, fixed cone; $C_3 p^2$ + ray.
18	"	"	" " 2 rays + ray of contact, surfaces touch quadric.
19	"	"	2 nodes; $C_3 pq + pq$; (C.).
20	"	"	binode p, node q; $C_2 p_f q$ + ray$_f$ + pq.
21	"	"	node p, fixed cone, node q; 2 rays + line of contact pq, surfaces touch plane.
22	"	"	node p, fixed cone, binode q; ray + line of osculation pq, surfaces touch plane.
23	"	1 point of contact.	no node; C_4; (C., L., S.).
24	"	"	node; $C_4 p^2$.
25	"	"	binode; $C_3 p_f p_v$ + ray$_f$.
26	"	"	node, fixed cone; 4 rays.

No.	Order.	Isolated points in H.	Multiple points; curves in H; (authorities).
27	3–5	6 points.	nodal line; (C., S.)
28	3–6	3 points.	no node; $C_1 + C_1$ of contact; (C., L., S.).
29	"	"	node; plane C_3; (C.).
30	"	"	" C_2 + ray.
31	"	"	" C_1 + ray of contact; (S.).
32	"	"	" ray of osculation, surfaces touch quadric.
33	"	"	binode; $C_2 p_f$ + ray$_v$.
34	"	"	node, fixed cone; $C_3 p^2$.
35	"	"	" " ray + ray of contact, surfaces touch quadric.
36	"	"	2 nodes; $C_2 p + pq$; (C.).
37	"	"	binode p, node q; $C_2 p_f q + pq$.
38	"	"	" " 2 rays$_f$ + pq.
39	"	"	node p, fixed cone, node q; ray + line of contact pq, surfaces touch plane.
40	"	"	node p, fixed cone, binode q; line of osculation pq, surfaces touch plane.
41	"	1 point of contact, 1 point.	no node; $C_1 + C_1 + C_1$; (L., S., Bonicelli).
42	"	" "	node; $C_3 p$.
43	"	" "	" C_1 + 2 rays.
44	"	" "	binode; $C_3 p_f p_v$.
45	"	" "	" $C_2 p_f$ + ray$_f$.
46	"	" "	" ray$_v$ + ray$_f$ of contact, surfaces touch quadric.
47	"	" "	node, fixed cone; 3 rays.
48	"	" "	2 nodes; $C_1 q + pq$ + ray.
49	"	" "	binode p, node q in fixed plane; ray$_v$ + line of contact pq, surfaces touch plane.
50	"	" "	3 nodes; $pq + pr + qr$; (C.).
51	"	1 point of osculation.	no node; C_3; (C., L., S.).
52	"	"	node; $C_3 p^2$.
53	3–7	4 points.	node, fixed cone; ray of contact, surfaces touch quadric.
54	"	"	2 nodes; $C_1 + pq$; (C.).
55	"	"	binode p, node q; ray$_f$ + pq.
56	"	"	node p, fixed cone, node q; line of contact pq, surfaces touch plane.
57	"	1 point of contact, 2 points.	node; C_2; (C.).
58	"	" "	binode; $C_2 p_f$.
59	"	" "	node, fixed cone; 2 rays.
60	"	2 points of contact.	node; C_1 + ray; (C.).
61	"	"	binode; ray$_f$ + ray$_v$.
62	"	"	" ray$_f$ of contact, surfaces touch quadric
63	"	"	2 nodes; ray + pq.
64	"	"	binode p, node q in fixed plane; line of contact pq, surfaces touch plane.
65	"	1 point of osculation, 1 point.	node; $C_2 p$; (C.).
66	"	" "	binode; 2 rays$_f$.
67	3–8	5 points.	binode p, node q; pq.
68	"	1 point of contact, 3 points.	node, fixed cone; ray.
69	"	2 points of contact, 1 point.	binode; ray$_v$.
70	"	1 point of osculation, 2 points.	node; C_1; (C.).
71	"	" "	binode; ray$_f$.
72	3–9	1 point of contact, 4 points.	node, fixed cone.
73	"	3 points of contact.	binode.
74	"	1 point of osculation, 3 points.	"
75	"	1 point of contact of 3rd order, 2 points.	node; (C., S., Beloch).

REFERENCES.

(See foot-note, p. 203.)

Beloch, *Ann. di Mat.*, (3) XVI, p. 64.

Berry, *Camb. Phil. Trans.* Cases of No. 1 are used in Vol. XIX, p. 281; of No. 2, XIX, p. 255; of No. 5, XIX, pp. 275, 282; XX, pp. 109, 111; of No. 7, XVIII, p. 334; XIX, p. 279; of No. 9, XIX, p. 281.

Bonicelli, *Giorn. di Mat.*, XL, p. 184.

Cayley, *Coll. Math. Papers*, VII, p. 229.

Del Pezzo, *Rendic. R. Acc. Napoli*, (3) II, p. 288. Cases of Nos. 3, 7 are included in § 28, and of No. 5 in § 3.

De Paolis, *Rendic. R. Acc. Linc.*, (4) I, pp. 735, 754. Cases of No. 2 are included in Cl. I, 2, II, 1, III, 2 (1°, 2°), 3 (1°), 4 (1°, 3°); of No. 3 in I, 1 (1°, 2°); of No. 5 in III, 1, 3 (2°).

Kantor, AMER. J. OF MATH., XIX, p. 1.

Noether, *Math. Ann.*, III, p 547.

VII.

All the quartic homaloidal families of Nos. 6–11 have multiple lines. Those of No. 11 have a fundamental system consisting of a triple straight line, three generators and a point; they are described by Cremona (*Ann. di Mat.*, 2 (V), p. 158). Those of No. 6 have a double conic and simple quintic; they are described by Sturm (*loc. cit.*, p. 371).

The $C_5 p^2$ of No. 7 meets four rays, pc, pd, pe, pf, and with these makes up the total intersection of ϕ with a cubic cone. The line l on ϕ which meets the remaining rays pa, pb is a 3-secant of C_5 and with it makes up the total intersection of ϕ with a quadric, which is the locus of l as ϕ varies within the homaloidal family. If ϕ contains any one other point of l, it contains the whole line; *i. e.*, all the points of l present the same condition to ϕ and correspond to the same point in the second space, which is therefore a point of a fundamental line of the quartic family (ϕ_4'); this fundamental line is simple because l is linear, and linear because ϕ contains only one line l. The same argument applied to the rays $pc, \ldots, pf$ shows that (ϕ_4') also has a fundamental simple quartic. Now if t is the fundamental isolated point on ϕ, the plane tpa meets ϕ in pa and a conic through t, p and three points on C_5; this conic presents only one condition to ϕ, and since ϕ contains two such conics, there is a fundamental nodal conic on (ϕ_4'). This with the simple quartic and straight line makes up the fixed part C_{13}' of the intersection of two of the quartic surfaces.

The $C_5 p_f^2 p_v$ of No. 8 meets one ray pf in the variable plane and with it makes up the total intersection of ϕ with a quadric cone. This ray gives rise to a fundamental straight line on (ϕ_4'); but all the cubic surfaces which contain pf have the same variable tangent plane at p, viz., that which contains pf and the tangent to the third branch of C_5 at p; the line on (ϕ_4') is therefore a line of contact. The three rays in the fixed plane give rise to a simple fundamental cubic on (ϕ_4'); and the planes through t and the remaining two rays to a nodal conic. The C_{13}' of fixed intersection of (ϕ_4') consists therefore of a conic counted four times, a straight line counted twice and a cubic.

In No. 9, the four rays which are not fixed give rise to a simple fundamental quartic on (ϕ_4'); the straight line on ϕ which meets the two fixed rays gives rise to a simple straight line; and the two planes through t and the fixed rays lead separately to nodal straight lines.

In No. 10, there are three rays at p which do not coincide in pq; these give rise to a simple cubic. The one other ray at q in the fixed plane gives rise to a simple straight line. The plane tpq meets ϕ in pq and a conic with a fixed tangent at p and also passing through t, q, which gives rise to a nodal straight line. All the cubic surfaces which pass through this conic have the same variable tangent plane at q, and the two sheets of ϕ_4' both touch the same fixed plane all along the nodal line, which counts eight times in the intersection. To make up the C_{13}' there is needed another simple straight line. This corresponds to pq; for pq counts three times in the intersection of ϕ, ϕ_1, and one of the three coincident lines may be thought of as separated from the others, and not meeting the variable C_4 of intersection of ϕ, ϕ_1, but constituting a fundamental line of the special kind.

VIII. *Composition.*

Of the 75 transformations enumerated above, there are 26 which can be compounded of quadro-quadric transformations (Nos. 1, 5, 11, 16, 20, 21, 22, 27, 30, 31, 33, 36, 37, 38, 41, 43, 45, 46, 48, 49, 50, 54, 55, 57, 58, 61); 16 others can be compounded of quadro-quadric transformations and others of the list (Nos. 9, 10, 17, 18, 26, 34, 35, 39, 40, 47, 59, 60, 63, 64, 65, 66); the remaining 33 are independent from the point of view of composition.

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